

AXIOMATISABILITY AND HARDNESS FOR UNIVERSAL HORN CLASSES OF HYPERGRAPHS

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ABSTRACT. We characterise finite axiomatisability and intractability of deciding membership for universal Horn classes generated by finite loop-free hypergraphs.

A *universal Horn class* is a class of model-theoretic structures of the same signature, closed under taking ultraproducts (P_u), direct products over nonempty families (P) and isomorphic copies of substructures (S); see [9, 17, 30, 32] for example. Equivalently they are classes axiomatisable by way of *universal Horn sentences*: universally quantified disjunctions $\alpha_1 \vee \dots \vee \alpha_k$, where each α_i is either an atomic formula of the language, or a negated atomic formula, and all but at most one of the α_i are negated. *Quasivarieties* are very closely related classes, differing from the universal Horn class definition only in that the trivial one-element structure (in which all relations are total) is automatically included; this corresponds to allowing the degenerate direct product over an empty family of structures.

Problems of axiomatisability for universal Horn classes and quasivarieties have a relatively long history. The starting point is perhaps Maltsev’s characterisation of semigroups embeddable in groups [28, 29], with subsequent developments in semigroup theory including Sapir [36], Margolis and Sapir [31], Jackson and Volkov [25]. There is also a wealth of literature within universal algebra and relational structures; see Gorbunov’s book [17], or the *Studia Logica* special issue [2] for many examples. An extra impetus for investigation of universal Horn classes comes from computational complexity. For example, the fixed template constraint satisfaction problem over a finite relational structure is the problem of deciding membership of relational structures in a certain universal Horn class [23]. Computational issues for universal Horn classes of relational structures also play a hidden role behind a number of examples demonstrating intractability of deciding membership of finite algebras in a finitely generated pseudovariety. Indeed, several of the relatively few known examples involve encoding a NP-complete universal Horn class membership problem into a pseudovariety membership problems. This is true for Szekely [37], Jackson and McKenzie [24] and [21] for example.

The present article concerns both axiomatisability and computational complexity for universal Horn classes of loop-free hypergraphs, and we are able to extend all of the known results for simple graphs. The characterisation of finitely axiomatisable universal Horn classes of finite simple graphs was given by Caicedo [10], by combining a probabilistic result of Erdős [14] with work of Nešetřil and Pultr [34]. In fact, Caicedo’s work covers any universal Horn class whose members have bounded chromatic number. After fixing a reasonable model-theoretic meaning to “hypergraph” we show that Caicedo’s classification may be extended to arbitrary loop-free

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hypergraphs. The precise statement depends on technicalities concerning how hyperedges are to be recorded as relations, but even without this it is possible to state an abridged version of the result as follows.

Theorem 1. *Let \mathcal{H} be a universal Horn class of hypergraphs without singleton hyperedges and with bounded chromatic number and hyperedge cardinality. If \mathcal{H} consists of disjoint unions of bipartite graphs—including the degenerate cases where there are no hyperedges—then \mathcal{H} has a finite axiomatisation in first order logic. In all other cases, \mathcal{H} has no finite axiomatisation in first order logic.*

In particular, \mathcal{H} can never have a finite axiomatisation if it contains a hypergraph with at least one hyperedge of arity more than 2. As in the case of Caicedo’s classification, the argument for nonfinite axiomatisability will again follow by probabilistic constructions (this time Erdős and Hajnal [15]), while we are able to show that the finitely axiomatisable case becomes almost completely degenerate. These results are found in Section 1. In Section 2 we extend another result in [10] by showing that there are continuum many universal Horn classes of hypergraphs. We apply a method of Bonato [7] to show that every interval in the homomorphism order on hypergraphs represents a continuum of universal Horn classes; this requires a new extension to a result of Nešetřil [33] on the density of the homomorphism order on hypergraphs. In Section 3 we turn to the question of axiomatisability amongst finite structures, a topic that has generated quite a lot of interest in finite model theory; see [35] for example. We are able to show that Theorem 1 continues to hold when restricted to finite structures only: this appears to be new even in the case of simple graphs, and shows that the model-theoretic SP-Preservation Theorem holds for classes of hypergraphs of bounded chromatic number and hyperedge cardinality. This is established using an Ehrenfeucht-Fraïssé game argument to observe a general lemma applying to any hereditary class of finite structures that is closed under certain disjoint unions. Finally, in Section 4 we observe an alternative path to the results of Sections 1 and 3 by application of the authors’ recent All or Nothing Theorem [18]; see Theorem 17 below. This approach has the advantage of adding complexity-theoretic hardness results for associated computational problems and avoiding the probabilistic constructions completely. However, the method depends on the All or Nothing Theorem, whose proof is substantially more involved than the direct constructions here.

1. HYPERGRAPHS

A *hypergraph* is a pair (V, E) , where V is a set—the *vertices*—and E is a set of non-empty subsets of V —the *hyperedges*. For $k \geq 1$, a *k-uniform hypergraph* is a hypergraph (V, E) where all hyperedges have exactly k elements.

Graphs coincide with hypergraphs in which all hyperedges have size at most 2, while *simple graphs* are the 2-uniform hypergraphs. Many graph-theoretic concepts extend to hypergraphs in reasonably obvious ways.

- An *n-cycle* is a sequence $v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$ alternating between distinct vertices v_0, \dots, v_{n-1} and distinct hyperedges e_0, \dots, e_{n-1} , such that $v_i \in e_i \cap e_{i+1}$ (with addition in the subscript taken modulo n).
- An *ℓ -colouring* of a hypergraph $\langle V; E \rangle$ is a function $\gamma : V \rightarrow \{0, 1, \dots, \ell-1\}$ such that $|\gamma(e)| \geq 2$ for each $e \in E$.
- the *chromatic number* χ of a hypergraph $\langle V; E \rangle$ is the smallest ℓ for which $\langle V; E \rangle$ is ℓ -colourable. In other words, the chromatic number is the smallest number of colours required to colour the vertices in such a way that no hyperedge is monochromatic.

- Two vertices z and w of V are *adjacent* if they belong to a common hyperedge, and are *connected* if there is a sequence $z = v_0, v_1, v_2, \dots, v_k = w$ of vertices of V in which v_{i-1} is adjacent to v_i , for $i = 1, 2, \dots, k$. A *connected hypergraph* is a hypergraph where every pair of vertices is connected.
- A *hyperforest* is a hypergraph without cycles, and *hypertree* is a connected hyperforest.

Let k be at least as large as the maximal hyperedge cardinality of a hypergraph $\mathbb{H} = (V, E)$. Then \mathbb{H} may be considered as a relational structure $\langle V; r_E \rangle$ with a single k -ary relation r_E by treating each hyperedge $\{v_1, \dots, v_\ell\}$ (where $\ell \leq k$) as the family of k -tuples $\{(v_{i_1}, \dots, v_{i_k}) \mid \{i_1, \dots, i_k\} = \{1, \dots, \ell\}\}$. We call such a structure a *k-hypergraph structure*.

Example 2. The simple graph \mathbb{K}_2 with edge $\{0, 1\}$. Treated as a 2-hypergraph structure, \mathbb{K}_2 is

$$\langle \{0, 1\}; \{(0, 1), (1, 0)\} \rangle.$$

As a 3-hypergraph structure \mathbb{K}_2 is

$$\langle \{0, 1\}; \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\} \rangle.$$

The class of all k -hypergraph structures is a universal Horn class, definable by the following *set-equivalence* universal Horn sentences.

Set equivalence: $(x_{i_1}, \dots, x_{i_k}) \in r \rightarrow (x_{j_1}, \dots, x_{j_k}) \in r$, if $\{x_{i_1}, \dots, x_{i_k}\} = \{x_{j_1}, \dots, x_{j_k}\}$.

The class of k -uniform hypergraphs (as model-theoretic structures) is a subclass of the k -hypergraph structures, defined by adjoining the following *uniformity* laws.

Uniformity: $(x_1, \dots, x_k) \in r \rightarrow x_i \neq x_j$ whenever $i \neq j$ and $i, j \in \{1, \dots, k\}$.

The class of loop-free hypergraphs (that is, with no singleton hyperedges) is a subclass obtained by adjoining the single universal Horn sentence $(x, \dots, x) \notin r$; clearly k -uniform hypergraphs are loop-free, except in the degenerate case of $k = 1$.

Remark 3. By default we choose the arity k to equal the maximal cardinality of any hyperedge in \mathbb{H} . Our methods cover the case where k is strictly larger than this, however the statement of results will be different. Example 2 illustrates the difference, as the homomorphism problem in the case of $k = 2$ is the tractable problem of graph 2-colouring but is NP-complete problem $+NAE3SAT$ when $k = 3$.

The notion of *induced subhypergraph* in the next definition coincides with the model-theoretic notion of substructure.

Definition 4. A hypergraph $\mathbb{G}' = \langle V'; E' \rangle$ is an induced subhypergraph of $\mathbb{G} = \langle V; E \rangle$ if $V' \subseteq V$ and $E' = \{e \cap V' \mid e \in E\}$.

The homomorphism notion also agrees with the model-theoretic homomorphism when both \mathbb{G} and \mathbb{G}' are considered as a k -hypergraph structures.

Definition 5. For any pair of hypergraphs $\mathbb{G} = (V, E)$ and $\mathbb{G}' = (V', E')$, a map $f : V \rightarrow V'$ is a homomorphism if for each $e \in E$, the set $f(e) = \{f(v) \mid v \in e\}$ is an element of E' .

As usual, $\mathbb{G} \rightarrow \mathbb{G}'$ will denote the statement “there exists a homomorphism from \mathbb{G} to \mathbb{G}' ” and $\mathbb{G} \not\rightarrow \mathbb{G}'$ will denote its negation.

Example 6. Let $\mathbb{K}_n^{(k)}$ denote the loop-free hypergraph on n points $\{0, 1, \dots, n-1\}$ and whose hyperedge set is the set of all subsets of $\{0, 1, \dots, n-1\}$ of size between 2 and k . Then a k -hypergraph structure \mathbb{H} is n -colourable if and only if $\mathbb{H} \rightarrow \mathbb{K}_n^{(k)}$ (as a k -hypergraph structure).

Note that the hypergraph $\mathbb{K}_n^{(2)}$ is the usual complete graph \mathbb{K}_n .

For a k -ary relation r , let us denote the *set closure* of r , denoted $\text{set}(r)$, to be the closure of r under applications of the set equivalence laws: in other words, if $(s_1, \dots, s_k) \in r$ is a tuple, then we add the tuple (s'_1, \dots, s'_k) to $\text{set}(r)$ whenever $\{s'_1, \dots, s'_k\} = \{s_1, \dots, s_k\}$. It is trivial that if $\mathbb{S} = \langle S; r^{\mathbb{S}} \rangle$ is a relational structure in the signature of a single k -ary relation r , then provided $r^{\mathbb{S}}$ has no constant tuples $(s, \dots, s) \in r^{\mathbb{S}}$, the structure $\mathbb{S} = \langle S; \text{set}(r^{\mathbb{S}}) \rangle$ is a k -hypergraph structure, and is a k -uniform hypergraph structure if $r^{\mathbb{S}}$ already satisfied the uniformity laws. We write $\text{set}(\mathbb{S})$ to denote the result of replacing $r^{\mathbb{S}}$ by $\text{set}(r^{\mathbb{S}})$. The following lemma is also trivial.

Lemma 7. *Let \mathbb{H} be a k -hypergraph structure and $\mathbb{S} = \langle S; r^{\mathbb{S}} \rangle$ be a relational structure with a single k -ary relation r . Then $\mathbb{S} \rightarrow \mathbb{H}$ if and only if $\text{set}(\mathbb{S}) \rightarrow \mathbb{H}$.*

The next theorem is essentially Theorem 5 of Feder and Vardi [16], except that their result is proved relative to the class of all relational structures rather than k -hypergraph structures. In the more general setting of [16], what we have written as \mathbb{H}_2^{\sharp} would not be a k -uniform hypergraph structure (but rather just some general relational structure of the same signature as \mathbb{H}_2) and the notion of cycle is more restrictive than the one we use. The statement we give in Theorem 8 follows immediately from [16, Theorem 5] after an application of the $\text{set}(\)$ operator and Lemma 7.

Theorem 8. *Fix any positive integer ℓ . Let \mathbb{H}_1 and \mathbb{H}_2 be k -hypergraph structures such that there is no homomorphism from \mathbb{H}_2 to \mathbb{H}_1 . Then there is a k -uniform hypergraph structure \mathbb{H}_2^{\sharp} such that*

- (1) $\mathbb{H}_2^{\sharp} \rightarrow \mathbb{H}_2$;
- (2) $\mathbb{H}_2^{\sharp} \not\rightarrow \mathbb{H}_1$;
- (3) *any cycle in \mathbb{H}_2^{\sharp} has size greater than ℓ .*

The following theorem is due to Erdős and Hajnal [15], but it follows immediately from Theorem 8 and Example 6, by choosing $\mathbb{H}_1 := \mathbb{K}_n^{(k)}$ and $\mathbb{H}_2 := \mathbb{K}_{n+1}^{(k)}$.

Theorem 9. *For any $k \geq 2$ and $\ell, n > 1$ there is a finite k -uniform hypergraph \mathbb{H} such that \mathbb{H} has no cycles of length less than ℓ and is not n -colourable.*

A hyperedge for which at most one vertex is contained in more than one hyperedge is called a *leaf*. A routine variation of the standard argument for 2-uniform hyperforests (that is, forests) shows that every finite hyperforest contains at least one leaf.

In the following we use the well known fact that a structure \mathbb{S} lies in the universal Horn class of some finite structure \mathbb{M} if and only if the following *separation conditions* hold:

- (SEP1) there exists a homomorphism ϕ from \mathbb{S} to \mathbb{M} ;
- (SEP2) for all $x, y \in S$ with $x \neq y$, there exists a homomorphism ψ from \mathbb{S} to \mathbb{M} satisfying $\psi(x) \neq \psi(y)$;
- (SEP3) for every relation r in the signature, with arity n , if $(s_1, \dots, s_n) \in S^n \setminus r^{\mathbb{S}}$, then there exists a homomorphism γ satisfying $(\gamma(s_1), \dots, \gamma(s_n)) \notin M^n \setminus r^{\mathbb{M}}$.

We mention that if \mathbb{S} is also finite then these conditions imply that \mathbb{S} is isomorphic to an induced substructure of a finite direct power of \mathbb{M} , indeed it is easy to prove that \mathbb{S} is isomorphic to a substructure of $\mathbb{M}^{\text{hom}(\mathbb{S}, \mathbb{M})}$, where $\text{hom}(\mathbb{S}, \mathbb{M})$ denotes the set of all homomorphisms from \mathbb{S} to \mathbb{M} , which is finite if \mathbb{S} and \mathbb{M} are finite.

Lemma 10. *Let $k \geq 3$ and $\mathbb{E}_k = \langle \{v_1, \dots, v_k\}; \{\{v_1, \dots, v_k\}\} \rangle$ be the hypergraph containing exactly one hyperedge. If \mathbb{E}_k is considered as a k -uniform hypergraph,*

then $\text{SP}(\mathbb{E}_k)$ contains all k -uniform hyperforests, with all finite k -uniform hyperforests lying in $\text{SP}_{\text{fin}}(\mathbb{E}_k)$.

Proof. Every relational structure embeds into an ultraproduct of its finite substructures, it suffices to prove the lemma in the case of finite hyperforests. Thus if we show that every finite k -uniform hyperforest lies in $\text{SP}(\mathbb{E}_k)$, then it follows that every k -uniform hyperforest lies in $\text{SP}(\mathbb{E}_k)$. Let $\mathbb{F} = \langle F; r^{\mathbb{F}} \rangle$ be a finite k -uniform hyperforest, with $r^{\mathbb{F}}$ the fundamental k -ary relation.

We proceed by induction on the number, n , of hyperedges of \mathbb{F} . For simplicity, we will assume that there are no isolated points as it is close to trivial to extend the separation conditions below to include these.

The base case with $n = 1$ is trivial, so assume that every k -uniform hyperforest with at most $n - 1$ hyperedges belongs to the class $\text{SP}(\mathbb{E}_k)$. Let $e = \{u_1, \dots, u_k\}$ be a leaf in \mathbb{F} . At most one vertex in e lies in any other hyperedge; if it exists denote it by u , which otherwise is a symbol not equal to the label of any vertex. Now let \mathbb{F}_{n-1} be the subhyperforest induced by removing the elements $\{u_1, \dots, u_k\} \setminus \{u\}$ from \mathbb{F} . By the induction hypothesis, we have $\mathbb{F}_{n-1} \in \text{SP}(\mathbb{E}_k)$ and so conditions (SEP1)–(SEP3) hold. We now show that $\mathbb{F} \in \text{SP}(\mathbb{E}_k)$. First we show that every homomorphism $\phi: \mathbb{F}_{n-1} \rightarrow \mathbb{E}_k$ extends to a homomorphism $\phi^+: \mathbb{F} \rightarrow \mathbb{E}_k$, giving (SEP1). Simply define $\phi^+(v) := \phi(v)$ for all $v \in \mathbb{F}_{n-1}$ (in particular u is sent to $\phi(u)$) and send each element in $\{u_1, \dots, u_k\} \setminus \{u\}$ to a different element of $\{v_1, \dots, v_k\} \setminus \phi(u)$. When $u \notin F$, this simply means we map $\{u_1, \dots, u_k\}$ onto $\{v_1, \dots, v_k\}$, giving $k!$ possible choices for ϕ^+ . When $u \in F$ there are $(k - 1)!$ choices for ϕ^+ .

Now let $e' = (w_1, \dots, w_k) \notin r^{\mathbb{F}}$ be any non-hyperedge of \mathbb{F} (for verifying (SEP3)) and let $x, y \in F$ with $x \neq y$ (for verifying (SEP2)). There are two cases to consider.

Case 1: If $\{w_1, \dots, w_k\}$ is a subset of F_{n-1} , then (SEP3) in the case of \mathbb{F}_{n-1} guarantees the existence of a homomorphism $\gamma: \mathbb{F}_{n-1} \rightarrow \mathbb{E}_k$ mapping e' strictly into $\{v_1, \dots, v_k\}$ (that is, to a non-hyperedge of \mathbb{E}_k). Then $\gamma^+: \mathbb{F} \rightarrow \mathbb{E}_k$ is the desired homomorphism for (SEP3). The same technique applies if the pair x, y with $x \neq y$ both lie in F_{n-1} , giving (SEP2).

Case 2: If $e' = (w_1, \dots, w_k)$ contains an element w_j not in F_{n-1} , then w_j is an element of $\{u_1, \dots, u_k\} \setminus \{u\}$. If $|\{w_1, \dots, w_k\}| < k$ then any homomorphism from \mathbb{F} to \mathbb{E}_k will fail to map $\{w_1, \dots, w_k\}$ onto $\{v_1, \dots, v_k\}$: since there exists a homomorphism ϕ from \mathbb{F}_{n-1} by (SEP1), the homomorphism ϕ^+ completes the argument for (SEP3) in this subcase.

Now assume that $|\{w_1, \dots, w_k\}| = k$, and observe that since $\{w_1, \dots, w_k\}$ is not a hyperedge of \mathbb{F} , it cannot be equal to the hyperedge $\{u_1, \dots, u_k\}$, and so contains at least one element from F_{n-1} other than u . Without loss of generality we may assume that w_1 is such an element. Note that $w_1 \neq w_j$ because $w_j \notin F_{n-1}$ by assumption. Fix any homomorphism $\phi: \mathbb{F}_{n-1} \rightarrow \mathbb{E}_k$ separating u from w_1 (which exists because \mathbb{F}_{n-1} satisfies (SEP2)) and define a homomorphism ϕ' from \mathbb{F} to \mathbb{E}_k in the following way. For $a \in F_{n-1}$, define $\phi'(a) := \phi(a)$ and define $\phi'(w_j) := \phi(w_1)$. Finally, let ϕ' send each element in $\{u_1, \dots, u_k\} \setminus \{u, w_j\}$ to a different element of $\{v_1, \dots, v_k\} \setminus \{\phi'(u), \phi'(w_j)\}$. Clearly, the map ϕ' is a homomorphism, and it maps e' to a non-hyperedge since $\phi'(w_1) = \phi'(w_j)$ implies that $\{\phi'(w_1), \dots, \phi'(w_k)\} \subsetneq \{v_1, \dots, v_k\}$. Thus (SEP3) holds.

To separate the pair $x \neq y$ when at least one of x, y is not in F_{n-1} , simply take any homomorphism $\phi: \mathbb{F}_{n-1} \rightarrow \mathbb{E}_k$ and note that a large number of the $(k - 1)!$ choices for ϕ^+ will separate x from y , establishing (SEP2). \square

Lemma 11. *Let $k > \ell > 1$ and $\mathbb{E} = \langle \{v_1, \dots, v_\ell\}; \{\{v_1, \dots, v_\ell\}\} \rangle$ be a hypergraph with exactly one hyperedge. Then if \mathbb{E} is treated as a k -hypergraph structure, the SP -class of \mathbb{E} includes the k -uniform hypergraph $\mathbb{E}_k := \langle \{u_1, \dots, u_k\}; \{\{u_1, \dots, u_k\}\} \rangle$.*

Proof. The homomorphisms from \mathbb{E}_k to \mathbb{E} coincide with the surjective maps from $\{u_1, \dots, u_k\}$ onto $\{v_1, \dots, v_\ell\}$. It is trivial that such maps exist (SEP1) and that any pair of points may be separated by a suitable map, given that $\ell \geq 2$ (SEP2). For (SEP3), a non-hyperedge of \mathbb{E}_k as a k -hypergraph structure is any k -tuple that has a repeat, say, $(u_{i_1}, \dots, u_{i_k})$ with $|\{u_{i_1}, \dots, u_{i_k}\}| = j$ for some $j < k$. If $j < \ell$ then every homomorphism maps $(u_{i_1}, \dots, u_{i_k})$ to a non-hyperedge. If $j \geq \ell$, then map $\{u_{i_1}, \dots, u_{i_k}\}$ onto $\{v_1, \dots, v_{\ell-1}\}$ and all remaining $k - j$ elements of $\{u_1, \dots, u_k\}$ onto $\{v_\ell\}$. \square

Consider the following SP-classes generated by a single hypergraph.

- Let $\mathbb{G}_1 = \langle \{1\}; \emptyset \rangle$ be the edgeless hypergraph on one vertex and let $\mathcal{Q}_{1(k)} = \text{SP}(\mathbb{G}_1)$, where \mathbb{G}_1 is treated as a k -hypergraph structure.
- Let $\mathbb{G}_2 = \langle \{1, 2\}; \emptyset \rangle$ be the edgeless hypergraph on two vertices and let $\mathcal{Q}_{2(k)} = \text{SP}(\mathbb{G}_2)$, where \mathbb{G}_2 is treated as a k -hypergraph structure.

Proof of Theorem 1. The following argument applies whenever \mathcal{K} is a class of loopfree hypergraphs of finite bounded hyperedge cardinality c and $k \geq c$; in the theorem statement, the class \mathcal{K} is $\text{SP}(\mathcal{K})$. If all members of \mathcal{K} have no hyperedges, then the universal Horn class generated by \mathcal{K} is equal to either $\mathcal{Q}_{1(k)}$ or $\mathcal{Q}_{2(k)}$. Now assume that \mathcal{K} contains a hypergraph with at least one hyperedge. The case where $k = c = 2$ is covered by Caicedo [10], so now assume that $k > 2$. Let \mathbb{H} be a hypergraph in \mathcal{K} containing a hyperedge e . Assume that e has minimal cardinality amongst the hyperedges of \mathbb{H} , and let \mathbb{E} denote the induced substructure on the elements of e , which consists of a single hyperedge e and lies in $\text{S}(\mathcal{K})$. As $k \geq 3$ we find by Lemma 11 that the singleton hyperedge k -uniform tree \mathbb{E}_k lies in $\text{SP}(\mathbb{E}) \subseteq \text{SP}(\mathcal{K})$. We will show that $\text{SP}(\mathcal{K})$ is not definable by any universal sentence by showing that for every $n \in \mathbb{N}$ there exists a k -uniform hypergraph \mathbb{U}_n such that the following properties hold.

- The structure \mathbb{U}_n is not in $\text{SP}(\mathcal{K})$.
- Every n -generated substructure of \mathbb{U}_n belongs to $\text{SP}(\mathbb{E}) \subseteq \text{SP}(\mathcal{K})$.

For $n \in \mathbb{N}$, Theorem 9 shows that there exists a finite k -uniform hypergraph \mathbb{U}_n with chromatic number strictly greater than that of \mathcal{K} and has no cycles of length less than $n + 1$. This necessarily places \mathbb{U}_n outside of $\text{SP}(\mathcal{K})$, as there are no homomorphisms from \mathbb{U}_n into any member of \mathcal{K} . However, an n -element induced substructure of \mathbb{U}_n is a k -uniform hyperforest, so lies in $\text{SP}(\mathbb{E}_k) \subseteq \text{SP}(\mathbb{E}) \subseteq \text{SP}(\mathcal{K})$, by Lemma 10. \square

Remark 12. We may also extend a result of Trotta from the class of simple graphs to the class of hypergraphs. Trotta [38, Theorem 2.4] showed that a simple graph is *standard* in the sense of Clark et al. [12] if and only if it either has no edges or consists only of disjoint unions of isolated points and single edge graphs. A version of Theorem 8 is used (via a construction from [13]) to show nonstandardness for any graph not equal to a disjoint union of complete bipartite graphs; see the proof of Theorem 3.9 in [38]. An identical argument for hypergraphs, shows that for $k \geq 3$, a k -hypergraph structure is standard if and only if it has no hyperedges. This also positively answers Problem 3 of [13] in the particular case of hypergraphs.

2. UNIVERSAL HORN CLASS LATTICES ARE CONTINUUM IN CARDINALITY

It is shown in Caicedo [10] that there are continuum many universal Horn classes of graphs. The argument has an easy adaptation to the present setting, but we instead follow a substantial extension of Caicedo's result proved by Bonato [7]: any interval in the homomorphism order on simple graphs (above the bipartite graphs)

contains continuum many universal Horn classes. Indeed, Bonato's very short argument shows that it suffices to show that intervals in the homomorphism order satisfy a density property. We mention that a density result corresponding to that cited by Bonato is known for hypergraphs—Nešetřil [33, Theorem 1.4]—however the proof there makes intrinsic use of hypergraphs of increasingly large hyperedge cardinality and so is not available here. In the proof of the following theorem we find an alternative proof of [33, Theorem 1.4] involving bounded hyperedge cardinality. As usual, we write $\mathbb{A} \rightarrow \mathbb{B}$ to denote the existence of a homomorphism from \mathbb{A} to \mathbb{B} .

Theorem 13. *Let \mathbb{G}_1 and \mathbb{G}_2 be finite k -hypergraph structures, both containing at least one hyperedge. For $i = 1, 2$, let \mathcal{U}_i denote the universal Horn class of all k -hypergraph structures admitting a homomorphism into \mathbb{G}_i . If $\mathbb{G}_1 \rightarrow \mathbb{G}_2$ but $\mathbb{G}_2 \not\rightarrow \mathbb{G}_1$ (equivalently, $\mathbb{G}_1 \in \mathcal{U}_2$ but $\mathbb{G}_2 \notin \mathcal{U}_1$), then there is a continuum of universal Horn classes between \mathcal{U}_1 and \mathcal{U}_2 .*

Proof. The argument of Bonato (in the proof of [7, Proposition 6]) applies immediately, provided we can show that the homomorphism order is dense between \mathcal{U}_1 and \mathcal{U}_2 . It suffices to show that there is a hypergraph \mathbb{H} lying strictly between \mathbb{G}_1 and \mathbb{G}_2 in the homomorphism order.

Because both \mathbb{G}_1 and \mathbb{G}_2 contain a hyperedge, it follows by Lemmas 10 and 11 (and (SEP1)) that all hyperforests admit a homomorphism into both \mathbb{G}_1 and \mathbb{G}_2 . Then the property $\mathbb{G}_2 \not\rightarrow \mathbb{G}_1$ shows that \mathbb{G}_2 does not admit a homomorphism into any hyperforest. Let \mathbb{G}_2^\sharp be the k -uniform hyperforest shown to exist in Theorem 8, with $\ell := |\mathbb{G}_2| + 1$, and let \mathbb{H} be the k -hypergraph structure $\mathbb{G}_2^\sharp \cup \mathbb{G}_1$. Then $\mathbb{H} \rightarrow \mathbb{G}_2$. Also, $\mathbb{G}_1 \rightarrow \mathbb{H}$ but $\mathbb{H} \not\rightarrow \mathbb{G}_1$. Thus, it remains to show that $\mathbb{G}_2 \not\rightarrow \mathbb{H}$. Now, at least one component of \mathbb{G}_2 does not homomorphically map into \mathbb{G}_1 , by assumption. To complete the proof, assume for contradiction that this component homomorphically maps into the \mathbb{G}_2^\sharp component of \mathbb{H} . By property (3) of Theorem 8, this component maps into a sub-hypertree of \mathbb{G}_2^\sharp , contradicting the fact that \mathbb{G}_2 does not have a homomorphism into any hyperforest. \square

3. AXIOMATISABILITY AT THE FINITE LEVEL

Let \mathcal{H} be an SP_{fin} -closed class of *finite* hypergraphs of bounded chromatic number (and hyperedge cardinality). The proof of Theorem 1 shows that unless \mathcal{H} consists only of disjoint unions of complete bipartite graphs, no finite set of universal Horn sentences can axiomatise \mathcal{H} amongst finite structures. A classical model-theoretic intuition (namely, the *SP-Preservation Theorem*; see McNulty [32]) would then imply that no first order sentence can define \mathcal{H} . In the restriction to finite structures however, there is no completely general SP-Preservation Theorem—see [13, Example 4.3]—though the possibility of such a result remains an open problem in the case of relational signatures; see [3, Problem 1] and [35, §2.4.2]. In this section we provide an argument that shows that the intuition is nevertheless correct in the case of hypergraphs: \mathcal{H} cannot be defined by any first order sentence at the finite level.

We prove a more general result, deducing the finite level version of Theorem 1 as a corollary. For any relational structure \mathbb{A} , we let $\overline{\mathbb{A}}$ be the graph on the same underlying set A , with edge relation obtained by placing an undirected edge between $a, b \in A$ whenever a and b appear together in the tuple of one of the relations of \mathbb{A} . When \mathbb{A} is a graph we have $\overline{\mathbb{A}} = \mathbb{A}$. Define the distance $d_{\mathbb{A}}(a, b)$ between two vertices a, b in \mathbb{A} to be the length of the shortest path of edges between a and b in $\overline{\mathbb{A}}$. Note that the distance may be infinite, which we denote by $d(a, b) = \infty$. When $a = b$ the distance $d(a, b)$ is 0. Let the n -ball $B_n(a)$ of a in \mathbb{A} be the set

$\{x \in A \mid d(x, a) \leq n\}$ and let $\mathbb{B}_n(a)$ denote the induced substructure of \mathbb{A} on $B_n(a)$. Note that the distance of any $b \in B_n(a)$ from a in $\mathbb{B}_n(a)$ remains equal to the distance from b to a in \mathbb{A} , but in general the distance between two elements of $B_n(a)$ distinct from a may be larger in $\mathbb{B}_n(a)$ than in \mathbb{A} . The following easy observation generalises this.

Observation 14. Let b, c be elements of an n -ball $B_n(a)$ in \mathbb{A} lying at distance j and j' from a respectively. If the distance δ from b to c in \mathbb{A} is at most $2n - j - j'$, then the distance $d_{\mathbb{B}_n(a)}(b, c)$ from b to c in $\mathbb{B}_n(a)$ is also δ .

Proof. Consider a path from b to c in $\overline{\mathbb{A}}$ of length δ . The first $n - j$ elements are distance at most $n - j + j = n$ from a , and the final $n - j'$ are distance at most $n - j' + j'$ from a . Thus all lie in $B_n(a)$ showing that the distance from b to c is δ in $\mathbb{B}_n(a)$ as well. \square

The *boundary* of an n -ball $\mathbb{B}_n(a)$ is the set of elements that are distance exactly n from a . Note that the n -ball $\mathbb{B}_n(a)$ can have empty boundary, such as if $n \geq 1$ and a is an isolated point.

Theorem 15. Let \mathcal{K} be an \mathbf{S} -closed class of finite structures of some relational signature such that for all n there exists a finite structure \mathbb{S}_n with the following properties:

- $\mathbb{S}_n \notin \mathcal{K}$;
- The disjoint union of any finite number of copies of n -balls in \mathbb{S}_n lies in \mathcal{K} .

Then \mathcal{K} cannot be defined amongst finite structures by any first order sentence.

Proof. We use a standard Ehrenfeucht-Fraïssé game argument: see Libkin [27]. For each k , let n be any integer greater than 2^{k+1} and let \mathbb{H}_k consist of the disjoint union of k copies of every n -ball $\mathbb{B}_n(a)$, for every $a \in \mathbb{S}_n$. Let \mathbb{G}_k denote the disjoint union of \mathbb{H}_k with \mathbb{S}_n . The second condition on \mathcal{K} trivially shows that $\mathbb{H}_k \in \mathcal{K}$. Because \mathcal{K} is \mathbf{S} -closed, the complement class to \mathcal{K} is closed under extensions and contains \mathbb{S}_n , by the second condition on \mathcal{K} . Thus $\mathbb{G}_k \notin \mathcal{K}$. We make frequent reference to the boundaries of n -ball components, and to the \mathbb{S}_n component, which we define to have no boundary.

We show how Duplicator has a winning strategy against Spoiler in a k -round Ehrenfeucht-Fraïssé game on the pair $\mathbb{G}_k, \mathbb{H}_k$. After $i \in \{0, 1, \dots, k\}$ rounds of the game, the players have selected points g_1, \dots, g_i from \mathbb{G}_k and h_1, \dots, h_i from \mathbb{H}_k , and Duplicator has not lost if the induced substructures on these points are isomorphic. At each round $i \in \{0, 1, \dots, k\}$, we will say that the distance between two elements x and x' in \mathbb{G}_k or \mathbb{H}_k is large_i if $d(x, x') \geq 2^{k-i+1}$. The basic idea is that whenever two points x, x' are distance at least large_i , then a selection of any third point will be at least large_{i+1} from one of x and x' (this follows because $\text{large}_i / 2 = 2^{k-i+1} / 2 = 2^{k-(i+1)+1} = \text{large}_{i+1}$) and that at the end of the game (when $i = k$) the value of large_k is greater than 1. For similar arguments, see Libkin [27, Chapter 3].

It is convenient to fix some isomorphisms between any two copies of an n -ball, and also between each copy of each n -ball $\mathbb{B}_n(a)$ component in either of \mathbb{G}_k or \mathbb{H}_k and the actual substructure $\mathbb{B}_n(a)$ of the \mathbb{S}_n component. Our strategy makes reference to these isomorphisms. When Duplicator makes a move in response to Spoiler, she will first decide which component to play in—as determined by distances between elements—and once this is chosen, select the appropriate corresponding element—as determined by the fixed isomorphism. Throughout the proof, we refer to “corresponding element” rather than make explicit reference to the fixed isomorphisms.

We will show, inductively, that Duplicator can not only maintain partial isomorphism but also preserve the following conditions at each of the rounds $i \in \{0, \dots, k\}$. For any $0 < \ell, j < i$:

- (1) the element h_ℓ in \mathbb{H}_k is a corresponding element of g_ℓ in \mathbb{G}_k ;
- (2) if $d_{\mathbb{G}_k}(g_\ell, g_j) < 2^{k-i+1}$, then $d_{\mathbb{H}_k}(h_\ell, h_j) = d_{\mathbb{G}_k}(g_\ell, g_j)$;
- (3) if $d_{\mathbb{G}_k}(g_\ell, g_j) \geq 2^{k-i+1}$, then $d_{\mathbb{H}_k}(h_\ell, h_j) \geq 2^{k-i+1}$;
- (4) for $\ell < 2^{k-i+1}$, the point g_j is of distance ℓ from a boundary if and only if h_j is of distance ℓ from a boundary.

The base case holds vacuously. For the induction step, suppose that Duplicator has maintained isomorphism and the four conditions to the completion of round i . We assume by default that Spoiler is making his $(i+1)^{\text{st}}$ move in \mathbb{G}_k , but note at key points how a similar argument would cover the case where his move is made in \mathbb{H}_k .

If Spoiler's selection for g_{i+1} is equal to some previously played element g_ℓ , where $\ell \leq i$, then Duplicator's response should be h_ℓ . Now assume that Spoiler selects an element not previously played.

Case 1: Any previously played element is distance greater than or equal to large_{i+1} from g_{i+1} .

Case 1(a): Spoiler chose g_{i+1} from the \mathbb{S}_n component of \mathbb{G}_k . In this case, Duplicator selects a copy of the ball $\mathbb{B}_n(g_{i+1})$ that has no previously played points in it: after round i there are at least $k-i$ unplayed copies remaining in \mathbb{H}_k . To maintain the hypotheses, Duplicator can select h_{i+1} to be the element corresponding to g_{i+1} .

Case 1(a) does not occur if Spoiler is selecting in \mathbb{H}_k .

Case 1(b): Spoiler chose g_{i+1} in one of the n -ball components, a copy of $\mathbb{B}_n(a)$, where a is some element in \mathbb{S}_n . Again, Duplicator finds an unused copy of $\mathbb{B}_n(a)$ in \mathbb{H}_k and selects h_{i+1} as the element corresponding to g_{i+1} . All comparative distances are large_{i+1} for both h_{i+1} and g_{i+1} , so the hypotheses are maintained. A symmetric argument applies when Spoiler is selecting in \mathbb{H}_k .

Case 2. There exists some previously played element g_ℓ ($\ell < i$) that is distance $d < \text{large}_{i+1}$ from g_{i+1} . Let \mathbb{B} denote the n -ball component of \mathbb{H}_k containing h_ℓ : it is isomorphic to some specific n -ball $\mathbb{B}_n(a)$ of \mathbb{S}_n , for some a .

Case 2(a). Spoiler chose g_{i+1} from an n -ball component of \mathbb{G}_k . Then g_ℓ lies in this same n -ball component, and by Condition (1) on g_ℓ and h_ℓ , this component is isomorphic to \mathbb{B} and the choice of h_{i+1} to correspond to g_{i+1} is guaranteed. Moreover, all comparative distances are identically equal or at least large_{i+1} for h_{i+1} as for g_{i+1} , so the hypotheses are maintained. A technicality here is if \mathbb{B} contains some element h_j for which g_j lies in the \mathbb{S}_n component so that $d_{\mathbb{G}_k}(g_{i+1}, g_j) = \infty \geq \text{large}_{i+1}$. But then $d_{\mathbb{G}_k}(g_\ell, g_j) = \infty$ also, so that $d(h_\ell, h_j) \geq \text{large}_i$, and then the property $d(h_{i+1}, h_\ell) < \text{large}_{i+1}$ implies $d(h_{i+1}, h_j) \geq \text{large}_{i+1}$ by the triangle inequality. In the dual to Case 2(a), a symmetric argument applies when Spoiler has selected h_{i+1} near some h_ℓ for which g_ℓ lies in an n -ball component of \mathbb{G}_k .

Case 2(b). Spoiler chooses g_{i+1} from the \mathbb{S}_n component. We will show that an element corresponding to g_{i+1} exists in \mathbb{B} , and that if Duplicator selects it as h_{i+1} , then the hypotheses are maintained.

We first show that g_{i+1} is contained in $B_n(a)$, the ball within \mathbb{S}_n isomorphic to the component containing h_ℓ . Now, there is no boundary in the \mathbb{S}_n component, so Condition (4) implies that h_ℓ is at least large_ℓ from the boundary of \mathbb{B} . Then the distance ϵ from h_ℓ to the centre of \mathbb{B} (the element corresponding to a) is at most $n - \text{large}_\ell$. Now the distance from g_ℓ to the point a in \mathbb{S}_k is exactly ϵ also, as h_ℓ corresponds to g_ℓ under the fixed isomorphism from \mathbb{B} to $\mathbb{B}_n(a)$. Hence the distance from g_{i+1} to a is at most $n - \text{large}_\ell + \text{large}_{i+1} \leq n - \text{large}_{i+1} < n$, so that g_{i+1} lies within the n -ball $\mathbb{B}_n(a)$ and a corresponding element h_{i+1} from \mathbb{B} can be selected.

Moreover h_{i+1} lies at least large_{i+1} from the boundary, so that both Conditions (1) and (4) hold. Observation 14 now shows that $d_{\mathbb{H}_k}(h_{i+1}, h_\ell) = d_{\mathbb{G}_k}(g_{i+1}, g_\ell)$ as well. In the dual case where Spoiler is choosing h_{i+1} near h_ℓ in \mathbb{H}_k and g_ℓ lies in the \mathbb{S}_n component, then the choice of g_{i+1} by Duplicator is immediate: use the fixed isomorphism from the component \mathbb{B} to $\mathbb{B}_n(a)$, with all issues relating to distances now identical to the case just detailed.

It now remains to verify that Conditions (2) and (3) are maintained for g_{i+1} in comparison to any other element g_j with $j \neq \ell$ and $j \leq i$.

Case 2(b)(i): If g_j is distance strictly less than large_i from g_ℓ , then Condition (2) of the hypothesis tells us that $d_{\mathbb{H}_k}(h_\ell, h_j) = d_{\mathbb{G}_k}(g_\ell, g_j)$. By the triangle inequality, the distance from g_{i+1} to g_j in \mathbb{G}_k is at most $d_{\mathbb{G}_k}(g_{i+1}, g_\ell) + d_{\mathbb{G}_k}(g_\ell, g_j) \leq \text{large}_{i+1} + \text{large}_i = 2n - (n - \text{large}_{i+1}) - (n - \text{large}_i)$ because $n > \text{large}_0$ and $j \neq \ell$ implies $i \geq 1$. Then Observation 14 shows that $d_{\mathbb{H}_k}(h_{i+1}, h_j) = d_{\mathbb{G}_k}(g_{i+1}, g_j)$ and both Condition (2) and (3) are maintained in this case for g_j .

Case 2(b)(ii): If g_j is distance greater than or equal to large_i from g_ℓ , then condition (1) of the hypothesis tells us that the distance of h_j from h_ℓ in \mathbb{H}_k is also at least large_i . Recall that $d_{\mathbb{H}_k}(h_\ell, h_{i+1}) = d_{\mathbb{G}_k}(g_\ell, g_{i+1}) \leq \text{large}_{i+1}$. Then the triangle inequality and the property $\text{large}_{i+1} = \text{large}_i/2$ imply that both $d_{\mathbb{G}_k}(g_{i+1}, g_j)$ and $d_{\mathbb{H}_k}(h_{i+1}, h_j)$ are at least large_{i+1} , showing that Conditions (2) and (3) are again maintained.

Finally we note that these conditions imply that the map $g_j \mapsto h_j$ is an isomorphism from the induced substructure on $\{g_1, \dots, g_{i+1}\}$ to $\{h_1, \dots, h_{i+1}\}$. Conditions (2) and (3) show that this function is a bijection. Assume that $(g_{i_1}, \dots, g_{i_k}) \in r$ is some hyperedge in the induced substructure on $\{g_1, \dots, g_{i+1}\}$. Then all distances between elements of g_{i_1}, \dots, g_{i_k} are at most 1. Hence, by Condition (2), the same is true for h_{i_1}, \dots, h_{i_k} . Hence all of h_{i_1}, \dots, h_{i_k} lie in the same n -ball component \mathbb{B} . Also, each h_{i_j} is a corresponding element to g_{i_j} , under the one fixed isomorphism from \mathbb{B} . Because this is an isomorphism, the tuple $(h_{i_1}, \dots, h_{i_k})$ lies in r within \mathbb{H}_k , as required. \square

Corollary 16. *Let \mathcal{H} be an SP_{fin} -closed class of loop-free k -hypergraph structures of bounded chromatic number. If $k = 2$ and \mathcal{H} contains a graph that is not a disjoint union of complete bipartite graphs, or if $k > 2$ and at least one member of \mathcal{H} has a hyperedge, then \mathcal{H} has no finite axiomatisation in first order logic amongst finite structures.*

Proof. Assume that $k = 2$ and \mathcal{H} contains a graph that is not a disjoint union of complete bipartite graphs, or $k > 2$ and at least one member of \mathcal{H} has a hyperedge. Note that \mathcal{H} coincides with the finite members of the universal Horn class $\text{SPP}_u(\mathcal{H})$. Theorem 9 shows that there is a k -uniform hypergraph \mathbb{V}_n not in $\text{SPP}_u(\mathcal{H})$ but whose cycles have length greater than $2n$. Then an n -ball in \mathbb{V}_n is a hyperforest. A disjoint union of hyperforests is still a hyperforest, and hyperforests lie in $\text{SPP}_u(\mathcal{H})$: in the case of $k = 2$, this is shown by Caicedo [10, Lemma 2], while the $k \geq 2$ case follows from Lemmas 10 and 11 above. Then Theorem 15 implies that \mathcal{H} is not definable amongst finite structures by any first order sentence. \square

4. HARDNESS

A well known result of Hell and Nešetřil [19] states that for a finite simple graph \mathbb{G} , if \mathbb{G} is bipartite then \mathbb{G} -colourability of finite graphs can be decided in polynomial time, but otherwise is NP-complete. The same dichotomy was recently established by the authors for universal Horn classes generated by finite simple graphs (with the same boundary of tractability). In this section we show how to use this to provide an alternative path to Corollary 16 in the case of a universal

Horn class generated by a finite loop-free hypergraph. The basic idea is that if a class of structures can be defined in first order logic amongst finite structures, then it cannot be NP-complete with respect to first order reductions—this follows from the known strict containment in $\text{AC}^0 \subsetneq \text{L} \subseteq \text{NP}$; see Immerman [20].

4.1. Background concepts. We begin with some basic concepts relating to the algebraic method in constraint satisfaction problem complexity. We give only the bare necessities for the arguments we need; see [4, 22, 26] for further background information on these concepts and their relationship to constraint satisfaction problems.

For any relational structure \mathbb{A} , we let $\text{CSP}(\mathbb{A})$ denote the computational problem of deciding if an input finite structure admits a homomorphism into \mathbb{A} (the *constraint satisfaction problem* over \mathbb{A} , or the \mathbb{A} -*colourability problem*), while $\text{QMEM}(\mathbb{A})$ is the computational problem of deciding if an input finite structure lies in the quasivariety of \mathbb{A} : this is almost identical to the problem of deciding membership in the universal Horn class of \mathbb{A} , as the universal Horn class and quasivariety differ by at most the one-element total structure, which has no impact on computational complexity, nor on the possible definability of the classes in first order logic.

A *polymorphism* is a homomorphism $f: \mathbb{A}^n \rightarrow \mathbb{A}$, where \mathbb{A}^n is the n^{th} direct power of \mathbb{A} . The polymorphism f is said to be *cyclic* if it satisfies the equation $f(x_0, x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, x_0)$ for all $x_0, \dots, x_{n-1} \in A$, and idempotent if it satisfies $f(x, \dots, x) = x$ for all $x \in A$. It is known that if r is a relation definable on \mathbb{A} by a $\exists\wedge$ formula (a conjunction of atomic formulæ, with some variables existentially quantified), and $\langle A; r \rangle$ has no cyclic polymorphism, then \mathbb{A} has no cyclic polymorphism.

4.2. Hardness and nonfinite axiomatisability. A fundamental contribution of Bulatov, Jeavons and Krokhin [8] was to show that if a finite relational structure \mathbb{A} has no proper retracts and fails a particular special condition on its polymorphisms, then $\text{CSP}(\mathbb{A})$ is NP-complete. Using the results of Barto and Kozik [5, Theorem 4.1] and then Chen and Larose [11, Lemma 6.4], the special condition can be stated as: there exists a cyclic polymorphism. For our purposes we will use the following equivalent condition, also from [5, Theorem 4.1]: for all primes $p > |A|$ there is a cyclic polymorphism of \mathbb{A} arity p . The authors' All or Nothing Theorem [18, Theorem 5.2] shows that the result of [8] can be transferred to the membership problem for the quasivariety (and universal Horn class) of \mathbb{A} : if \mathbb{A} has no cyclic polymorphism, then $\text{QMEM}(\mathbb{A})$ is NP-complete with respect to first order reductions. The main result of this section is a corollary of this.

Theorem 17. *Let $k \geq c$ and $\mathbb{H} = \langle H; r \rangle$ be a finite loop-free k -hypergraph structure with maximal hyperedge cardinality c .*

- (Hell and Nešetřil [19], Ham and Jackson [18].) *If $k = c = 2$ or has no hyperedges at all, then $\text{CSP}(\mathbb{H})$ and $\text{QMEM}(\mathbb{H})$ are tractable if and only if \mathbb{H} is bipartite.*
- *Otherwise (that is, $k > 2$ and there is a hyperedge of some cardinality $c \neq 0$), then $\text{CSP}(\mathbb{H})$ and $\text{QMEM}(\mathbb{H})$ are NP-complete with respect to first order reductions and neither can be defined by a first order sentence at the finite level.*

Proof. The first statement is trivial when there are no hyperedges at all. When $k = c = 2$, then the $\text{CSP}(\mathbb{H})$ case is directly from [19] and the $\text{QMEM}(\mathbb{H})$ case is directly from [18]. Now, assume that $k > 2$ and \mathbb{H} has a hyperedge e .

Consider a hyperedge e of minimal cardinality $d \leq c$. First assume that $d > 2$ and consider the binary relation \sim defined from r by the formula

$$\exists x_3 \dots \exists x_d (x_1, x_2, x_3, \dots, x_{d-1}, \overbrace{x_d, \dots, x_d}^{k-d+1}) \in r$$

in free variables x_1, x_2 . The formula $(x_1, x_2, x_3, \dots, x_{d-1}, \overbrace{x_d, \dots, x_d}^{k-d+1}) \in r$ interprets in all hyperedges of cardinality d and no others, so that \sim is the graph consisting of d -cliques on each hyperedge of cardinality d . As $d > 2$ this graph is not bipartite, hence has no cyclic polymorphism by Barto, Kozik and Niven [6]. Thus \mathbb{H} has no cyclic polymorphism, as required.

Now assume that $d = 2$, and let the elements in the hyperedge e be denoted $0, 1$. We show that for any prime $p > |H|$, there is no cyclic polymorphism of arity p . Assume for contradiction that such a p -ary polymorphism exists. Let s_0, \dots, s_{p-1} be a sequence in $\{0, 1\}^p$ with the property that cyclically there is no run of k consecutive 0s, nor k consecutive 1s. Such sequences are very easily seen to exist, given that $k > 2$. Let $a \in H$ be the value of $f(s_0, \dots, s_{p-1})$. Because f is cyclic we have the following equalities:

$$\begin{array}{cccccc} f(s_0, & s_1, & s_2, & \dots, & s_{p-1}) & = & a \\ f(s_1, & s_2, & s_3, & \dots, & s_0) & = & a \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \\ f(s_{k-1}, & s_k, & s_{k+1}, & \dots, & s_{k-2}) & = & a \end{array}$$

Because there is no run of k consecutive values in s_0, \dots, s_{p-1} (treated cyclically), the tuples (s_i, \dots, s_{i+k-1}) forming columns on the left of the equalities lie in the fundamental relation r on \mathbb{H} . Hence as f is a polymorphism, the constant tuple (a, \dots, a) is in r . But this contradicts the assumption that \mathbb{H} was loop-free. So no cyclic polymorphism of arity p exists, as required. \square

Remark 18. The All or Nothing Theorem of [18] actually shows a stronger result than what is stated in Theorem 17. Whenever Theorem 17 states NP-completeness of $\text{QMEM}(\mathbb{H})$ the following holds: any class of finite hypergraphs \mathcal{K} has NP-hard membership problem provided its members admit homomorphisms into \mathbb{H} and that $\text{SP}_{\text{fin}}(\mathbb{H}) \subseteq \mathcal{K}$.

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